# SOME TYPES OF LINEAR COMBINATIONS OF WEYL-HEISENBERG WAVE PACKET FRAMES OVER LOCAL FIELDS OF POSITIVE CHARACTERISTIC

LALIT K. VASHISHT

Abstract. In this paper, we present necessary and sufficient conditions for some types of linear combination of frame elements (wave packet) to be a frame for $L^2(\mathbb{K})$, where $\mathbb{K}$ is a local field of positive characteristic.

## 1. Introduction

Let $\mathcal{H}$ be a separable real or complex Hilbert space (finite or infinite dimensional) with inner product $\langle .,. \rangle$. A countable sequence $\{f_k\}_{k\in I} \subset \mathcal{H}$ is called a *frame* (or Hilbert frame) for $\mathcal{H}$ if, there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{k\in I} |\langle f, f_k\rangle|^2 \leq B\|f\|^2 \text{ for all } f \in \mathcal{H}. \tag{1.1}$$

The positive constants $A$ and $B$ are called the *lower* and *upper frame bounds* of the frame $\{f_k\}_{k\in I}$, respectively. They are not unique. The inequality in 1.1 is called the *frame inequality* of the frame. If $\{f_k\}_{k\in I}$ satisfies the upper inequality in 1.1, then we say that $\{f_k\}_{k\in I}$ is a *Bessel sequence* in $\mathcal{H}$ with Bessel bound $B$.

Frames in Hilbert spaces were originally introduced by Duffin and Schaeffer [23] in 1952 in the context of non-harmonic Fourier series and popularized in 1986 by Daubechies, Grossmann, and Meyer [19]. Frames are basis-like building blocks that span a vector space but allow for linear dependency, which can be used to reduce noise, find sparse representations, spherical codes, compressed sensing, signal processing, and wavelet analysis etc., see [9].

A type of sequence which is obtained by combined action of three operators and have frame properties, introduced first time by Cordoba and Fefferman [17] by applying certain collection of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. This sequence is called *wave packet*. Later, Labate et al. [29] adopted the same expression to describe, more generally, any collection of functions which are obtained by applying the same operations to a finite family of functions in $L^2(\mathbb{R})$. More precisely, Gabor systems, wavelet systems and the Fourier transform of wavelet systems are special cases of wave packet systems. Lacey and Thiele [30, 31] gave applications of wave packet systems in boundedness of the Hilbert transforms. The wave packet systems have been studied by several authors, see [13, 14, 18, 21, 26, 27].

Recently many mathematicians and engineers generalized the frame and wavelet theory in various directions, see [9, 11, 12] and references therein. Benedetto and Benedetto [8] developed a wavelet theory for local fields and related groups. Behera and Jahan [2] - [7] developed wavelets on local

2010 *Mathematics Subject Classification.* 42C15; 42C30; 42C40.

*Key words and phrases.* Frames, Wave Packet Frames, Perturbation.

fields of positive characteristic. In this paper discuss some frame properties of wave packet system which is natural generalization of wavelets over local fields of positive characteristic.

1.1. **Background on Local Fields:** Let $\mathbb{K}$ be a field and a topological space. Then $\mathbb{K}$ is called a locally compact field or a local field if both $\mathbb{K}^+$ and $\mathbb{K}^-$ are locally compact abelian groups, where $\mathbb{K}^+$ and $\mathbb{K}^-$ denote the additive and multiplicative groups of $\mathbb{K}$, respectively. Any field endowed with the discrete topology is a local field. The only connected local fields are $\mathbb{R}$ and $\mathbb{C}$. Disconnected local fields are, in fact, totally disconnected. Every local field comes equipped with a canonical absolute value which defines its topology. Let $dx$ be the Haar measure for $\mathbb{K}^+$. For any non-zero $\alpha \in \mathbb{K}$, $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We say that $|\alpha|$ is the absolute value or valuation of $|\alpha|$.

The map $x \to |x|$ has the following properties:

$(i)$ $|x| = 0$ if and only if $x = 0$.
$(ii)$ $|xy| = |x||y|$ for all $x, y \in \mathbb{K}$.
$(iii)$ $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{K}$.

Note that $|x+y| = \max\{|x|, |y|\}$, if $|x| \neq |y|$. The set $\mathfrak{D} = \{x : |x| \leq 1\}$ is called the ring of integers in $\mathbb{K}$. It is the unique maximal compact subring of $K$. The set $\mathfrak{B} = \{x : |x| < 1\}$ is called the prime ideal in $\mathbb{K}$. Note that the prime ideal in $\mathbb{K}$ is the unique maximal ideal in $\mathfrak{D}$. It is principal and prime. The set of values $|x|$ as $x$ varies over $\mathbb{K}$ is a discrete set of the form $\{s^k : k \in \mathbb{Z}\} \cup \{0\}$ for some $s > 0$. So there is an element of $\mathfrak{B}$ of maximal absolute value. A prime element of $\mathbb{K}$ an element of maximum absolute value in $\mathfrak{B}$. The ring of integers $\mathfrak{D}$ is compact and open, so $\mathfrak{B}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime $p$ and $c \in \mathbb{N}$. If $K$ is a local field, then there is a nontrivial, unitary, continuous character $\chi$ on $\mathbb{K}^+$. It can be proved that $\mathbb{K}^+$ is self dual. Let $\chi$ be a fixed character on $\mathbb{K}^+$ that is trivial on $\mathfrak{D}$ but is nontrivial on $\mathfrak{B}^{-1}$. We can find such a character by starting with any nontrivial character and rescaling. We will define such a character for a local field of positive characteristic. For $y \in \mathbb{K}$, define $\chi_y(x) = \chi(yx)$, $x \in \mathbb{K}$.

Let $E$ be a measurable subset of $\mathbb{K}$ and let $|E| = \int_{\mathbb{K}} \mathbf{1}_E(x)$, where $\mathbf{1}_E$ is the characteristic function of $E$ and $dx$ is the Haar measure of $\mathbb{K}$ normalized so that $|\mathfrak{D}| = 1$. Then, $|\mathfrak{B}| = q^{-1}$ and $|\mathfrak{p}| = q^{-1}$. Therefore, if $x \neq 0$ and $x \in K$, then $|x| = q^k$ for some $k \in \mathbb{Z}$.

To impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$, first we note that $\Delta = \mathfrak{D}/\mathfrak{B} \cong GF(q)$ and and $GF(q)$ is a $c$-dimensional vector space over the field $GF(p)$. Therefore, we can find a set $\mathcal{A} = \{1 = \epsilon_0, \epsilon_1, \epsilon_2, ....\epsilon_{c-1}\} \subset \mathcal{D}^*$ such that span$\mathcal{A} \cong GF(q)$.

Every $n \in \mathbb{N}_o = \mathbb{N} \bigcup \{0\}$ with $0 \leq n \leq q - 1$ can be expressed as

$$n = a_0 + a_1 p + .... + a_{c-1} p^{c-1}, \quad 0 \leq a_k \leq q1, \quad k = 0, 1, ...., c - 1.$$

Let

$$u(n) = a_0 + a_1\epsilon_1 + .... + a_{c-1}\epsilon_{c-1})\mathfrak{p}^{-1}.$$

Then, $\{u(n) : n = 0, 1, ...., q - 1\}$ is a complete set of coset representatives of $\mathfrak{D}$ in $\mathfrak{B}^{-1}$. For a non-negative integer $n$ we write

$$n = b_0 + b_1 q + b_2 q^2 + .... + b_s q^s, \quad 0 \leq b_k \leq q - 1, \quad k = 0, 1, 2, ....s,$$

and define

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + .... + u(b_s)\mathfrak{p}^{-s}.$$

The function $u(.)$ in general, not additive. But, for all $r \geq 0, k \geq 0$ and $0 \leq s \leq q^k - 1$, we have

$$u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s).$$

**Definition 1.1.** Let $\mathbb{K}$ be a local field of characteristic $p > 0$. The character $\chi$ on $\mathbb{K}$ is defined as

$$\chi(\epsilon_\mu \mathfrak{p}^{-j}) = \begin{cases} exp(2\pi i/p), \ \mu = 0 \text{ and } j = 1 \\ 1, \ \mu = 1, 2..., c-1 \text{ or } j \neq 1, \end{cases}$$

where $\{1 = \epsilon_0, \epsilon_1, \epsilon_2, ....\epsilon_{c-1}\}$ be same as above.

Note that $\chi$ is trivial on $D$ but nontrivial on $\mathfrak{B}^{-1}$. Since $K = \bigcup_{j\in\mathbb{Z}} \mathfrak{p}^{-j}\mathfrak{D}$, so we can regard $\mathfrak{p}^{-1}$ as the dilation (note that $|\mathfrak{p}^{-1}| = q$). Furthermore, $\{u(n) : n \in N_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in $\mathbb{K}$, the set $\{u(n) : n \in N_0\}$ can be treated as the translation set. Next, we recall operators which are used in wave packet system. Let $a, b \in \mathbb{K}$. Define operators $T_a, \ E_b, \ D_\mathfrak{p}$ on $L^2(\mathbb{K})$ by

$$\begin{aligned} &T_a f(t) = f(t-a), \ f \in L^2(\mathbb{K}) \quad \text{(Translation by } a) \\ &E_b f(t) = \chi(bt) f(t), \ f \in L^2(\mathbb{K}) \quad \text{(Modulation by } b) \\ &D_\mathfrak{p} f(t) = q^{\frac{1}{2}} f(\mathfrak{p}^{-1} t) \quad \text{(Dilation by prime } \mathfrak{p}), \end{aligned}$$

As in case of standard wavelet and Gabor analysis these three class of operators play an important role in local field setting. For any $f \in L^2(\mathbb{K})$, we have

$$(\widehat{D_\mathfrak{p} f})(\gamma) = q^{\frac{-1}{2}} \widehat{f}(\mathfrak{p}\gamma), \quad \widehat{(T_a f)}(\gamma) = \overline{\chi_a(\gamma)} \widehat{f}(\gamma)$$

1.2. **Wave Packet System over local fields:** The wave packet system over local field of positive characteristic is natural generalization of wavelet structure on local fields. Let $\psi \in L^2(\mathbb{K})$. A system $\mathcal{W}(\psi) \equiv \left\{ D_{\mathfrak{p}^j} T_{u(k)a} E_{u(m)b} \psi : j, k, m \in \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0 \right\}$ is called a *Irregular Weyl-Heisenberg wave packet frame* (in short, *wave packet frame*) for $L^2(\mathbb{K})$. A frame for $L^2(\mathbb{K})$ of the form $\mathcal{W}(\psi)$ is called an *irregular Weyl-Heisenberg wave packet frame* (or *wave packet frame*. That if there exists positive scalars $A_\psi \leq B_\psi$ such that

$$A_\psi \|f\|^2 \leq \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \sum_{m\in\mathbb{N}_0} |\langle f, D_{\mathfrak{p}^j} T_{u(k)a} E_{u(m)b} \psi \rangle|^2 \leq B_\psi \|f\|^2 \text{ for all } f \in L^2(\mathbb{K}). \tag{1.2}$$

1.3. **Motivation for the work.** In his paper [1], an approach to determine all frames for separable Hilbert spaces is discussed. More precisely, starting from a frame $\{f_k\}_{k=1}^\infty$ for a separable Hilbert space $\mathcal{H}$, Aldroubi considered the question: "which conditions on the scalars $\{u_{l,k}\}_{k=1}^\infty$ will imply that the vectors

$$\phi_l = \sum_{k=1}^\infty u_{l,k} f_k, \quad l \in \mathbb{N} \tag{1.3}$$

are well defined and constitute a frame for $\mathcal{H}$? Note that by the frame decomposition, it is clear that all frames for $\mathcal{H}$ can be constructed this way. Aldroubi proved necessary and sufficient conditions for $\{\phi_l\}_{l=1}^\infty$ to be a frame for $\mathcal{H}$ (including in terms of boundedness of the operator $U = \{u_{l,k}\}_{l,k=1}^\infty$ on $\ell^2(\mathbb{N})$).

Kaushik, Singh and Virender [28] introduced the notion of $WH$ packets with respect to a Gabor system $\{E_{mb}T_{na}\psi\}_{m,n\in\mathbb{Z}}$, $(\psi \in L^2(\mathbb{R}))$, which is defined by a linear commination of the form

$$\psi_{r,s} = \sum_{(j,k)\in\mathbb{I}_{r,s}} \alpha_{j,k} E_{bj} T_{ak}\psi,\ (r,s\in\mathbb{Z}), \tag{1.4}$$

where $\bigcup_{r,s\in\mathbb{Z}} \mathbb{I}_{r,s} = \mathbb{Z}\times\mathbb{Z}$, $\mathbb{I}_{r,s}\bigcap \mathbb{I}_{r',s'} = \emptyset$, $(r,s)\neq(r',s')$, for all $r,s,r',s'\in\mathbb{Z}$ and $\alpha_{j,k}$ are scalars. They proved necessary and sufficient conditions for $WH$ packets $\{\psi_{r,s}\}_{r,s\in\mathbb{Z}}$ to be frame for $L^2(\mathbb{R})$. This idea of linear combination given in (1.4) was extended to wave packet setting in [33], where some similar results (given in [28]) are proved. In this paper, we discuss some frame properties of linear combination of wave packet system over local field of positive characteristic.

1.4. **Outline of the paper.** Let $\mathcal{W}(\psi) \equiv \Big\{D_{\mathfrak{p}^j}T_{u(k)a}E_{u(m)b}\psi : j,k,m\in\mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0\Big\}$ be a wave packet system in $L^2(\mathbb{K})$.
Let

$$\psi_{r,s,t} = \sum_{(j,k,m)\in\mathbb{I}_{r,s,t}} \alpha_{j,k,m} D_{\mathfrak{p}^j}T_{u(k)a}E_{u(m)b}\psi,\ (r,s,t\in\mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0), \tag{1.5}$$

where $\bigcup_{r,s,t\in\mathbb{Z}} \mathbb{I}_{r,s,t} = \mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0$, $\mathbb{I}_{r,s,t}\bigcap\mathbb{I}_{r',s',t'} = \emptyset$, $(r,s,t)\neq(r',s',t')$, for all $r,s,t,r',s',t'\in\mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0$ and $\alpha_{j,k,m}$ are scalars. Then, in general, $\psi_{r,s,t}$ $(r,s,t\in\mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0)$ does not constitute a frame for $L^2(\mathbb{R})$ (even not well defined). We present some necessary and sufficient conditions for $\mathcal{L}(\psi)\equiv\{\psi_{r,s,t}\}_{r,s,t\in\mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0}$ to be a frame for $L^2(\mathbb{K})$.

## 2. Necessary and Sufficient Conditions

The first result extends [28, Theorem 3.5].

**Theorem 2.1.** *Let $\mathcal{W}(\psi)$ be a frame for $L^2(\mathbb{K})$. Let $\Theta : \ell^2(\mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0)\to\ell^2(\mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0)$ defines a bounded linear operator such that*

$$\Theta : \{\langle D_{\mathfrak{p}^j}T_{u(k)a}E_{u(m)b}\psi, f\rangle\}_{j,k,m\in\mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0} \to \{\langle\psi_{r,s,t}, f\rangle\}_{r,s,t\in\mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0},\quad f\in L^2(\mathbb{K}).$$

*Then, $\mathcal{L}(\psi)$ is a frame for $L^2(\mathbb{K})$ if and only if*

$$\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{N}_0}\sum_{t\in\mathbb{N}_0}|\langle\psi_{r,s,t}, f\rangle|^2 \geq \lambda \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0}|\langle D_{\mathfrak{p}^j}T_{u(k)a}E_{u(m)b}\psi, f\rangle|^2,\ \text{for all } f\in L^2(\mathbb{K}).$$

*Proof.* Similar to proof of [33, Theorem 2.4]. □

The following theorem provided sufficient conditions on scalars which appear in $\mathcal{L}(\psi)$, to a frame for $L^2(\mathbb{K})$. This generalizes [28, Theorem 3.4] in wave packet setting.

**Theorem 2.2.** *Assume that $\mathcal{W}(\psi)$ is a frame for $L^2(\mathbb{K})$. If $\sup\limits_{r,s,t\in\mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0}|\mathbb{I}_{r,s,t}|<\infty$, and $\{\alpha_{j,k,m}\}$ is positively confined sequence and*

$$\inf_{r,s,t\in\mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0}\left\{\sum_{(j,k,m)\in\mathbb{I}_{r,s,t}}|\alpha_{j,k,m}|^2 - \sum_{\substack{(j,k,m),(j',k',m')\in\mathbb{I}_{r,s,t}\\(j,k,m)\neq(j',k',m')}}|\alpha_{j,k,m}|\ |\alpha_{j',k',m'}|\right\} > 0,$$

*then $\mathcal{L}(\psi)$ is a frame for $L^2(\mathbb{R})$.*

*Proof.* Similar to proof of [33, Theorem 2.5]. □

Aldroubi proved sufficient conditions on scalars in his linear combination of frame vectors. This result was extended to wave packet setting in [33, Theorem 2.6]. This following theorem gives sufficient conditions on the scalars which similar to standard frames [11, Proposition 5.5.8] to wave packet frames on local fields.

**Theorem 2.3.** *Suppose that $\mathcal{W}(\psi)$ is a frame for $L^2(\mathbb{K})$ with frame bounds $A, B$ and for a fix $s \in Z$, let $\{\alpha_{s,j,k,m}\}_{j,k,m \in \mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0}$ be a square summable sequence. If*

$$\mu := \inf_{j,k,m \in \mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0} \left[ \sum_{s\in\mathbb{Z}} |\alpha_{s,j,k,m}|^2 - \sum_{\substack{j',k',m' \in \mathbb{Z} \\ (j,k,m)\neq(j',k',m')}} \left| \sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m} \overline{\alpha_{s,j',k',m'}} \right| \right] > 0,$$

$$\nu := \sup_{j,k,m \in \mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0} \sum_{j',k',m' \in \mathbb{Z}} \left| \sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m} \overline{\alpha_{s,j',k',m'}} \right| < \infty.$$

*Then, the vectors* $\sum\limits_{j,k,m \in \mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}_0} \alpha_{s,j,k,m} D_{\mathfrak{p}^j} T_{u(k)a} E_{u(m)b}\psi$ *constitute a frame for $L^2(\mathbb{K})$ with bounds $\mu A, \nu B$.*

*Proof.* Similar to proof of [33, Theorem 2.6]. □

It is well known that the finite sum of frames need not be a frame for the underlying space. Kaushik et al. [28] proved necessary and sufficient conditions for finite sum of Gabor frames. Similar results for finite sum of in wave packet setting can be found in [33]. The following result extends [28, Theorem 4.2] (in wave packet setting [33, Theorem 2.9]).

**Theorem 2.4.** *Let $\Big\{\mathcal{W}(\psi_l) : l = 1, 2, ....n\Big\}$ be a finite family wave packet frames for $L^2(\mathbb{K})$. Then, $\Big\{\sum\limits_{l=1}^{n} \alpha_l \mathcal{W}(\psi_l)\Big\}$ is a frame for $L^2(\mathbb{K})$ if and only if there exists $M_o > 0$ and some $p$ $(1 \leq p \leq n)$ such that*

$$M_o \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \sum_{m\in\mathbb{N}_0} |\langle D_{\mathfrak{p}^j} T_{u(k)a} E_{u(m)b}\psi_p, f\rangle|^2 \leq \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \sum_{m\in\mathbb{N}_0} \left| \left\langle \sum_{l=1}^{n} \alpha_l D_{\mathfrak{p}^j} T_{u(k)a} E_{u(m)b}\psi_l, f \right\rangle \right|^2, f \in L^2(\mathbb{K})$$

*where $\alpha_1, \alpha_2, ...., \alpha_n$ are scalars.*

*Proof.* Similar to proof of [33, Theorem 2.9]. □

The following theorem extends [28, Theorem 4.3].

**Theorem 2.5.** *Let $\Big\{\mathcal{W}(\psi_l) : l = 1, 2, ....n\Big\}$ be a finite family wave packet frames for $L^2(\mathbb{K})$ with frame bounds $A_l,\ B_l$ $(1 \leq l \leq n)$. Let $\alpha 1, \alpha_2, ...., \alpha_l$ be positive scalars. Then, $\Big\{\sum\limits_{l=1}^{n} \alpha_l \mathcal{W}(\psi_l)\Big\}$ is a frame for $L^2(\mathbb{K})$ provided*

$$\alpha_p A_p > \sum_{l=1,\ l\neq p}^{n} B_l + 2 \sum_{l,j=1,\ l\neq p, j\neq p, l\neq j}^{n} \alpha_l \overline{\alpha_j} \sqrt{B_l B_j} \tag{2.1}$$

*for some $p$, $(1 \leq p \leq n)$.*

*Proof.* Let $f \in L^2(\mathbb{K})$ be arbitrary.
Then

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0}\Big|\Big\langle\sum_{l=1}^{n}\alpha_n D_{\mathfrak{p}^j}T_{u(k)a}E_{u(m)b}\psi_l, f\Big\rangle\Big|^2\\&\leq n|\alpha_1|^2B_1\|f\|^2+n|\alpha_2|^2B_2\|f\|^2+....+n|\alpha_n|^2B_n\|f\|^2\\&=n\Big(\sum_{l=1}^{n}|\alpha_n|^2\Big)\|f\|^2.\end{aligned}$$

This gives upper frame inequality.
By hypothesis (2.1), we can show that

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0}\Big|\Big\langle\sum_{l=1}^{n}\alpha_n D_{\mathfrak{p}^j}T_{u(k)a}E_{u(m)b}\psi_l, f\Big\rangle\Big|^2\\&\geq\Big[\alpha_pA_p-\sum_{l=1,\ l\neq p}^{n}B_l+2\sum_{l,j=1,\ l\neq p,j\neq p,l\neq j}^{n}\alpha_l\overline{\alpha_j}\sqrt{B_lB_j}\Big]\|f\|^2,\ f\in L^2(\mathbb{K}).\end{aligned}$$

Hence $\left\{\sum_{l=1}^{n}\alpha_l\mathcal{W}(\psi_l)\right\}$ is a frame for $L^2(\mathbb{K})$ □

**Corollary 2.6.** *In Theorem 2.5, if* (2.1) *is replaced by*

$$\alpha_pA_p<\sum_{l=1}^{n}\alpha_l^2A_l-\sum_{s,t=1,s\neq t}\alpha_s\alpha_t\sqrt{B_sB_t},$$

*for some* $p$ $(1\leq p\leq n)$, *then* $\left\{\sum_{l=1}^{n}\alpha_l\mathcal{W}(\psi_l)\right\}$ *is a frame for* $L^2(\mathbb{K})$.

*Proof.* Similar to proof of [33, Theorem 2.10].

**Remark 2.7.** Let $\left\{\mathcal{W}(\psi_l):l=1,2,....n\right\}$ be a finite family wave packet frames for $L^2(\mathbb{K})$ with frame bounds $A_l,\ B_l$ $(1\leq l\leq n)$. If $\left\{\sum_{l=1}^{n}\alpha_l\mathcal{W}(\psi_l)\right\}$ is a frame for $L^2(\mathbb{K})$ with frame bounds $A_o$ and $B_o$. Then, $A_l,\ B_l$ and $A_o, B_o$ are related in the following way:

$$\sum_{l=1}^{n}\alpha_l^2A_l-\sum_{l,t=1,l\neq t}\alpha_l\alpha_t\sqrt{B_lB_t}\leq B_o$$

and

$$\sum_{l=1}^{n}\alpha_l^2B_l+\sum_{l,t=1,\neq t}\alpha_l\alpha_t\sqrt{B_lB_t}\geq A_o,$$

where square of scalars have standard meaning.

□

## References


[1] A. Aldroubi, Portraits of frames, *Proc. Amer. Math. Soc.*, 123 (1995), 1661–1668.

[2] B. Behera, Haar wavelets on the Lebesgue spaces of local fields of positive characteristic, *Colloq. Math.*, 136 (2014), 149–168.

[3] B. Behera and Q. Jahan, Wavelet packets and wavelet frame packets on local fields of positive characteristic, *J. Math. Anal. Appl.*, 395 (2012), 1–14.

[4] B. Behera and Q. Jahan, Multiresolution analysis on local fields and characterization of scaling functions, *Adv. Pure Appl. Math.*, 3 (2012), 181–202.

[5] B. Behera and Q. Jahan, Biorthogonal wavelets on local fields of positive characteristic, *Commun. Math. Anal.*, 15 (2013), 52–75.

[6] B. Behera and Q. Jahan, Characterization of wavelets and MRA wavelets on local fields of positive characteristic, *Collect. Math.*, 66 (2015), 33–53.

[7] B. Behera and Q. Jahan, Affine and quasi-affine frames on local fields of positive characteristic, preprint.

[8] J. J. Benedetto and R. L. Benedetto, A wavelet theory for local fields and related groups, *J. Geom. Anal.*, 14 (2004), 423–456.

[9] P. G. Casazza and G. Kutyniok, *Finite Frames: Theory and Applications*, Birkhäuser (2012).

[10] P. G. Casazza and N. Leonhard, Classes of finite equal norm Parseval frames, *Contemp. Math.*, 451 (2008), 11–13.

[11] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston (2002).

[12] O. Christensen, *Frames and Bases: An Introductory Course*, Birkhäuser, Boston (2008).

[13] O. Christensen and A. Rahimi, Frame properties of wave packet systems in $L^2(\mathbb{R}^d)$, *Adv. Compu. Math.*, 29 (2008), 101–111.

[14] O. Christensen and A. Rahimi, An introduction to wave packet Systems in $L^2(\mathbb{R})$, *Indian J. Indus. Appl. Math.*, 1 (2008), 42–57.

[15] O. Christensen, H. O. Kim and R. Y. Kim, On Parseval wavelet frames with two or three generators via the unitary extension principle, *Canad. Math. Bull.* , 57(2) (2014), 254–263.

[16] O. Christensen, H. O. Kim and R. Y. Kim, Extensions of Bessel sequences to dual pairs of frames, *Appl. Comput. Harmon. Anal.*, 34 (2013), 224–233.

[17] A. Cordoba and C. Fefferman, Wave packets and Fourier integral operators, *Comm. Partial Differential Equations*, 3 (11) (1978), 979–1005.

[18] W. Czaja, G. Kutyniok and D. Speegle, The geometry of sets of prameters of wave packets, *Appl. Comput. Harmon. Anal.,* 20 (2006), 108–125.

[19] I. Daubechies, A. Grossmann and Y. Meyer, Painless nonorthogonal expansions, *J. Math. Phys.*, 27 (1986), 1271–1283.

[20] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia (1992).

[21] Deepshikha and L. K. Vashisht, Extension of Bessel sequences to dual frames in Hilbert spaces, submitted.

[22] Deepshikha and L. K. Vashisht, Extension of Weyl-Heisenberg wave packet Bessel sequences to dual frames in $L^2(\mathbb{R})$, Preprint.

[23] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.*, 72 (1952), 341–366.

[24] D. Gabor, Theory of communications, *J. Inst. Elect. Eng.* (London), 93 (3) (1946), 429–457.

[25] D. Han, Dilations and completions for Gabor systems, *J. Fourier Anal. Appl.*, 15 (2)(2009), 201–217.

[26] E. Hernández, D. Labate and G. Weiss, A unified characterization of reproducing systems generated by a finite family II, *J. Geom. Anal.*, 12 (4) (2002), 615–662.

[27] E. Hernández, D. Labate, G. Weiss and E. Wilson, Oversampling, quasi-affine frames and wave packets, *Appl. Comput. Harmon. Anal.*, 16 (2004,) 111–147

[28] S. K. Kaushik, G. Singh and Virender, On $WH$ packets in $L^2(\mathbb{R})$, *Commun. Math. Appl.*, 3 (3) (2012), 333–344.

[29] D. Labate, G. Weiss and E. Wilson, An approach to the study of wave packet systems, *Contemp. Math.*, 345 (2004), 215–235.

[30] M. Lacey, C. Thiele, $L^p$ estimates on the bilinear Hilbert transform for $2 < p < \infty$, *Ann. Math.*, 146 (1997), 693–724.

[31] M. Lacey, C. Thiele, On Calderón's conjecture, *Ann. Math.*, 149 (1999), 475–496.

[32] D. F. Li and W. Sun, Expansion of frames to tight frames, *Acta. Math. Sin.* (Engl. Ser.), 25 (2009), 287–292.

[33] A. K. Sah and L. K. Vashisht, Irregular Weyl-Heisenberg wave packet frames in $L^2(\mathbb{R})$, *Bull. Sci. Math.*, 139 (1) (2015), 61–74.



L. K. Vashisht, Department of Mathematics, University of Delhi, Delhi-110007, India.

*E-mail address*: `lalitkvashisht@gmail.com`